

Instabilities of twisted strings

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Instabilities of twisted strings

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ABSTRACT: A linear stability analysis of *twisted* flux-tubes (strings) in an SU(2) semilocal theory — an Abelian-Higgs model with *two* charged scalar fields with a global SU(2) symmetry — is carried out. Here the twist refers to a relative phase between the two complex scalars (with linear dependence on, say, the z coordinate), and importantly it leads to a global current flowing along the the string. Such twisted strings bifurcate with the Abrikosov-Nielsen-Olesen (ANO) solution embedded in the semilocal theory. Our numerical investigations of the small fluctuation spectrum confirm previous results that twisted strings exhibit instabilities whose amplitudes grow exponentially in time. More precisely twisted strings with a single magnetic flux quantum admit a continuous family of unstable eigenmodes with harmonic z dependence, indexed by a wavenumber $k \in [-k_m, k_m]$. Carrying out a perturbative semi-analytic analysis of the bifurcation, it is found that the purely numerical results are very well reproduced. This way one obtains not only a good qualitative description of the twisted solutions themselves as well as of their instabilities, but also a quantitative description of the numerical results. Our semi-analytic results indicate that in close analogy to the known instability of the embedded ANO vortex a twisted string is also likely to expand in size caused by the spreading out of its magnetic flux.

KEYWORDS: Gauge Symmetry, Global Symmetries

Contents

1	The SU(2) semilocal theory and its vortices	4
1.1	The SU(2) semilocal theory	4
1.2	Vortex solutions	5
1.3	Bifurcation of the embedded ANO solution	6
2	Linear stability analysis	8
2.1	Perturbation analysis	8
2.2	Bifurcation analysis of the perturbation operator	13
2.3	Perturbative calculation of the eigenfunction corresponding to the instability	17
2.4	Properties of the instability modes	19
3	Conclusion	23
A	Details of the perturbation equations	23
A.1	The perturbation operator of the twisted vortices	24
A.2	The $\beta \rightarrow \infty$ limiting case	26
A.3	The expansion of the perturbation operator in the bifurcation parameter	27
B	Perturbations of the ANO vortices	28

Introduction

Topological defects, such as domain walls, strings and monopoles arise in many field theory models with spontaneous symmetry breaking. These objects are expected to be created during phase transitions, and are likely to play an important role in the early universe and more generally in cosmology [1]. Typically these defect solutions are stable because there is an infinite energy barrier separating them from the vacuum. An important class of defects is constituted by line-defects called cosmic strings. A cosmic string is typically a flux-tube, enclosing a certain number of magnetic flux quanta. In the plane orthogonal to its direction a cosmic string corresponds to a vortex solution. Vice versa, any vortex in the plane can be extended to a straight string in the orthogonal direction to the plane. The well-known Abrikosov-Nielsen-Olesen (ANO) vortex of the Abelian Higgs model serves as a prototype straight cosmic string. ANO vortices are characterized by an integer winding number, n , determining their magnetic flux, and by the mass ratio, $\beta = m_s^2/m_v^2$ (m_s resp. m_v denoting the mass of the scalar resp. vector fields).

Rather interesting, cosmic string-type defect solutions have been found in semilocal theories where there is no topological stability [2–4]. An important class of semilocal theories is provided by Abelian Higgs models with a suitably extended scalar sector, the

simplest one being two complex scalar fields admitting global $SU(2)$ symmetry. Semilocal models are quite interesting since both global and local symmetries are simultaneously broken; for a comprehensive review we refer to [5]. Abelian Higgs models with an extended scalar sector contain ANO-type vortices, which correspond to a simple embedding. Quite remarkably semilocal models exhibit stable vortices, despite the first homotopy group of the vacuum manifold being trivial [2, 3]. The linear stability of ANO vortices embedded into $SU(N)$ symmetric semilocal models has been examined in ref. [3], by computing their small fluctuation spectrum. It has been found that embedded ANO vortices are stable only if $\beta \leq 1$. For $\beta > 1$ there is a family of energy lowering eigenmodes, parameterized by the z -direction wave number, $k \in [-k_m, k_m]$, with the lowest lying eigenmode being z -independent. This homogeneous unstable mode can be seen to correspond to a “magnetic spreading” instability [3, 7]. The long time dynamics of the instability (i.e. the numerical solution of the full nonlinear field equations) has also been studied, in ref. [10]. There it is found that for $\beta > 1$ embedded ANO vortices undergo indeed a homogeneous expansion as expected from the analysis of ref. [3].

The case $\beta = 1$ is quite special, for here, instead of there being a unique (and stable) vortex solution satisfying the first order Bogomol’nyi equations [6], there is a continuous family of them having the same energy [3]. The Standard Model counterparts of the embedded ANO solutions are the Z-strings, which have been shown to be stable for $\sin^2 \theta_W \gtrsim 0.9$ [8].

Recently, a new family of current carrying, “twisted” string solutions have been discovered in the $SU(2)$ symmetric semilocal model [9]. These solutions are characterized by a relative phase difference, $\exp(i\omega z)$, between the two components of the scalar field, where z is the coordinate along the string and ω is the twist. Twisted strings exist only for $\beta > 1$ where the embedded ANO solutions are unstable. Quite remarkably the energy of twisted strings is lower than that of the embedded ones (as a matter of fact the energy per unit length of twisted strings is a monotonously decreasing function of the twist ω). The instability of the embedded ANO vortices for $\beta > 1$ can be seen to correspond to a bifurcation with twisted strings. Clearly the problem of stability of twisted strings is an important one.

A linear stability analysis of twisted strings has been presented by ref. [16], and a family of unstable modes has been found. More precisely for all values of the twist ω , the lowest eigenvalue belongs to a nonzero value of the wave number k of the eigenmode in the longitudinal direction. In particular there are no homogeneous (z -independent) negative energy eigenmodes. Noting the analogy between the harmonic z -dependence of the instability mode and the Plateau-Rayleigh instability in hydrodynamics (i.e. the fragmentation of a fluid stream into droplets), it has been argued in ref. [16] that the unstable eigenmode of twisted strings signals its breakup into small droplets.

In this paper, we present a detailed analysis of the stability of twisted vortices. Working in a different gauge than ref. [16] our work provides an independent check of the results of ref. [16]. Both ref. [16] and the present paper apply linearization analysis. In the present paper, following ref. [11], we work in the background field gauge. By solving the small fluctuation equations numerically for a large range of the parameters of the problem we confirm the numerical results of ref. [16], although we also find some small discrepancies

which do not affect, however, the validity of the numerical results. We also find that twisted strings possess a family of unstable modes, parametrized by their wave number in the z direction, $k \in [-k_m, k_m]$. These results are in agreement with those of ref. [16]. In addition to a full fledged numerical approach, we have also developed a semi-analytical description of the instability eigenmodes of twisted vortices for values of the twist near the bifurcation point. In our view this sheds some light on the nature of the instability of twisted strings. The description of the bifurcation also makes it possible to obtain the eigenvalues and eigenmodes of embedded strings as a deformation of those of twisted strings (with the deformation parameter being a function of the difference of the twist ω and the bifurcation point ω_b). We note that twisted strings with periodic boundary conditions are also unstable in the semilocal model.

Clearly a linear stability analysis is not sufficient to draw definite conclusions on the issue of the final state of twisted strings. There are nevertheless two important properties of the instabilities which can form the basis of some speculations on the long-time dynamics of twisted vortex instability. Firstly, not far from the bifurcation with the ANO vortices, the unstable modes of twisted vortices and those of the embedded ANO ones are very similar. The field component, which dominates the eigenmodes close to the one corresponding to the largest negative eigenvalue is nearly homogeneous. Secondly, independently of the absence of z -independent instability eigenmodes for twisted strings the z -independent instability mode of the embedded ANO vortices is still *an energy lowering* perturbation. In such a case, a sufficiently general initial perturbation (like a lump on the string, see subsection 2.4) overlaps with the eigenmode corresponding to the most rapidly growing ones, those being almost z -independent. Therefore we predict that twisted strings whose twist is close to the value at the bifurcation point start to expand nearly homogeneously very similarly to the instability of embedded ANO vortices. For twisted strings whose twist is far from the bifurcation value this expansion may become local, in the sense that expanded lumps may form on the strings. The long-time dynamics, however, is expected to be rather different for twisted strings as compared to the embedded ANO ones. This difference is due to the global current flowing in twisted strings. We have found that to first order in perturbation theory the current remains localized to the string. Unless the current is completely carried away by radiative effects which come from higher orders in perturbation theory, in contrast to the expansion of the embedded ANO strings [10], the expansion of twisted ones cannot go on indefinitely. It is also conceivable that the instability of static (or stationary) twisted strings signals that there are time dependent (oscillating) breather-like states. We also present some arguments, that twisted strings are unlikely to break up into small droplet-like configurations.

The outline of the paper is as follows: in section 1 we recapitulate the main characteristics of the SU(2) semilocal theory and its vortex solutions. In subsection 1.3 we give a semi-analytical description of the bifurcation of twisted vortices with the embedded ANO vortex. In section 2 we turn our attention to the stability problem of the twisted vortex solutions, and present the numerical results. In subsections 2.2 and 2.3 the behavior of the linearized equations near the bifurcation is studied. In subsection 2.4 the properties of the eigenmodes are analyzed, and the possible scenarios for the dynamics of the instability are presented. The technical details of the calculations of section 2 are relegated to appendix A.

In appendix B a summary of the perturbations of embedded ANO vortices is given.

1 The SU(2) semilocal theory and its vortices

1.1 The SU(2) semilocal theory

The 4 dimensional (4D) action of the SU(2) semilocal theory can be transformed by suitable rescalings, rendering the rescaled fields and coordinates to be dimensionless, and setting the charge of the scalar particle to unity, to the form:

$$S = \int d^4x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \Phi)^\dagger D^\mu \Phi - \frac{\beta}{2} (\Phi^\dagger \Phi - 1)^2 \right\}. \quad (1.1)$$

Where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $D_\mu \Phi = \partial_\mu \Phi - iA_\mu \Phi$, $\Phi^T = (\phi_1, \phi_2)$. The signature of the flat Minkowskian metric used here is $(+, -, -, -)$. Because of the spontaneous breaking of the U(1) gauge symmetry the physical spectrum contains two massive particles, a scalar and a vector whose mass ratio is given by $\beta = m_s^2/m_v^2$. The fields transform under the U(1) gauge symmetry as

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda(x), \quad \Phi \rightarrow e^{i\Lambda(x)} \Phi, \quad (1.2)$$

while the complex doublet, $\Phi^T = (\phi_1, \phi_2)$, transforms as the fundamental representation of the global SU(2) symmetry. The Euler-Lagrange equations following from the action (1.1) can be written as:

$$\partial^\rho F_{\rho\mu} = i\{(D_\mu \Phi)^\dagger \Phi - \Phi^\dagger D_\mu \Phi\}, \quad (1.3a)$$

$$D_\rho D^\rho \Phi = \beta(1 - |\Phi|^2)\Phi, \quad (1.3b)$$

where $|\Phi|^2 = \Phi^\dagger \Phi = |\phi_1|^2 + |\phi_2|^2$. Any solution of the Abelian-Higgs model with a single scalar field, (A_μ, ϕ_1) , can be embedded into the semilocal model simply by putting $(A_\mu, \Phi := \phi_1 \Phi_0)$ where Φ_0 is a constant SU(2) doublet of unit norm, $\Phi_0^\dagger \Phi_0 = 1$.

The SU(2) symmetric semilocal theory has the following conserved Noether currents

$$j_\mu^{\hat{a}} = -i \left((D_\mu \phi_a)^* T_{ab}^{\hat{a}} \phi_b - \phi_a^* T_{ab}^{\hat{a}} D_\mu \phi_b \right), \quad (1.4)$$

where $\hat{a} = (0, a)$ and $T^{\hat{a}} = (1, \tau^a)$ with τ^a being the Pauli matrices. The currents j_μ^a generate the global SU(2) while j_μ^0 the local U(1) symmetry.

We shall also consider the special theory corresponding to the $\beta \rightarrow \infty$ limit. In this limiting case, the scalar fields are constrained by $\Phi^\dagger \Phi \equiv 1$, and the limit theory is nothing but a gauged \mathbf{CP}^1 -model. The constraint can be taken into account by replacing the scalar potential in the Lagrangian of eq. (1.3) by a Lagrange multiplier term:

$$\mathcal{L}_\infty = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi_a)^* D^\mu \phi_a - \lambda(|\phi|^2 - 1). \quad (1.5)$$

The equations of motion in this case are very similar to the regular case,

$$\begin{aligned} D_\mu D^\mu \phi_a &= -\lambda \phi_a \\ \partial^\mu F_{\mu\nu} &= -i(\phi_c^* D_\nu \phi_c - \phi_c (D_\nu \phi_c)^*), \end{aligned} \quad (1.6)$$

with

$$\lambda = \phi_a^* D_\mu D^\mu \phi_a = \phi_a (D_\mu D^\mu \phi_a)^*. \quad (1.7)$$

1.2 Vortex solutions

To obtain twisted vortex solutions of the theory (1.1), one needs to write down the most general axially symmetric Ansatz [9]. Here, “axially symmetric” is meant in the general sense, that rotations around the symmetry axis, or translations in its direction can be compensated by a suitable gauge transformation [12].

Let us choose the $z = x_3$ coordinate along the symmetry axis. Choosing a suitable reference frame the configuration can be rendered static, which choice will be assumed from now on. The axially symmetric configuration has rotational symmetry in the (x_1, x_2) plane, and is then described by the following general Ansatz:

$$\begin{aligned} \phi_1(r, \vartheta, z) &= f_1(r)e^{in\vartheta}, \\ \phi_2(r, \vartheta, z) &= f_2(r)e^{im\vartheta}e^{i\omega z}, \\ A_\vartheta(r, \vartheta, z) &= na(r), \\ A_3(r, \vartheta, z) &= \omega a_3(r), \end{aligned} \tag{1.8}$$

with $A_0 = 0, A_r = 0$. The parameter ω , which describes the z dependence of the relative phase of the two scalar field components, is the twist.

In what follows we shall consider the simplest and most important class of configurations with $m = 0$ and with $n = 1$, but of course our analysis can be easily extended for the general case. The vortex profile functions in (1.8) obey the equations

$$\begin{aligned} \frac{1}{r}(ra'_3)' &= 2a_3|f|^2 - 2f_2^2, \\ r\left(\frac{a'}{r}\right)' &= 2f_1^2(a - 1) + 2f_2^2a, \\ \frac{1}{r}(rf'_1)' &= f_1\left[\frac{(1-a)^2n^2}{r^2} + \omega^2a_3^2 - \beta(1 - |f|^2)\right], \\ \frac{1}{r}(rf'_2)' &= f_2\left[\frac{(na)^2}{r^2} + \omega^2(1 - a_3)^2 - \beta(1 - |f|^2)\right]. \end{aligned} \tag{1.9}$$

Regularity at the origin $r = 0$ is ensured by the boundary conditions $a \rightarrow 0, f_1 \rightarrow 0, f_2 \rightarrow c, a_3 \rightarrow c'$, where c, c' are constants, while for $r \rightarrow \infty$ we demand $a \rightarrow 1, f_1 \rightarrow 1, f_2, a_3 \rightarrow 0$.

The embedded Abrikosov-Nielsen-Olesen (ANO) solutions correspond to $f_2 = a_3 = 0$ in eqs. (1.9). Besides these, new classes of vortex solutions have been discovered. First, ref. [3] found a one-parameter family of solutions for $\beta = 1$ with $f_2 \neq 0$ and $a_3 = 0$. The parameter of the vortices in this family of solutions can be identified with their width (denoting the parameter with ξ , the radial decay of the magnetic field is $1 - (r/\xi)^{-2}$). These solutions satisfy a Bogomol'nyi type energy bound [2], therefore their energy is degenerate, and they obey simpler, first order field equations.

Another one-parameter class of solutions has been discovered in [9] for $\beta > 1$ with both f_2 and $a_3 \neq 0$. These solutions are parameterized by the value of their twist, $0 < \omega < \omega_b(\beta)$. At $\omega = \omega_b$ these twisted vortices bifurcate with the embedded ANO vortex (for some numerical values of $\omega_b(\beta)$ for a range of β see ref. [9]). There is a similarity between this family of vortices and the one of ref. [3]. The parameter ω also characterizes the width of

vortices: a lower value of ω corresponds to a more diluted vortex with a slower radial decay of the magnetic field, and a higher value of the scalar condensate at the origin. (However, one should note, that the radial decay of these solutions is different from that of the $\beta = 1$ ones, as here f_2 decays as $\exp(-\omega r)$.)

An important property of twisted vortices is that there is a global current flowing along them. In the coordinate system fixed by the Ansatz (1.8), the current is the third isospin component of the semilocal current (1.4) in the z direction. The total current flowing along the string can thus be written as

$$I_3 = \int r dr d\theta j_3^3. \tag{1.10}$$

As the twist ω decreases, the current increases and the twisted vortex becomes more and more spread out [9]. It appears that the limit $\omega \rightarrow 0$ is somewhat singular, nevertheless in this limit twisted vortices converge pointwise to a configuration which is precisely the large width limit of the $\beta = 1$ semilocal vortices of ref. [3].

Another important property of twisted vortices is that they have lower energy than the embedded ANO solutions. This indicates that such twisted vortices may be preferred by physical processes over the embedded ANO vortices. In fact, for fixed β , the energy monotonously decreases as ω decreases, i.e. as we go further away from the bifurcation [9].

In the $\beta \rightarrow \infty$ limiting theory f_1 and f_2 are related by the constraint $f_1^2 + f_2^2 \equiv 1$. A convenient way of parameterizing them is

$$f_1 = \sin \theta, \quad f_2 = \cos \theta, \tag{1.11}$$

which yields the profile equations

$$\begin{aligned} \frac{1}{r} (ra'_3)' &= 2 [a_3 - \cos^2 \theta], \\ r \left(\frac{a'}{r} \right)' &= 2(a - \sin^2 \theta), \\ \frac{1}{r} (r\theta')' &= \frac{1}{2} \left[(2a_3 - 1)\omega^2 - \frac{n^2}{r^2}(2a - 1) \right] \sin(2\theta). \end{aligned} \tag{1.12}$$

In this case the embedded ANO solution has infinite energy, whereas the twisted vortices are of finite energy and they exist for all values of ω [9].

1.3 Bifurcation of the embedded ANO solution

It is by now well known [3] that the embedded ANO vortices are unstable to small perturbations of the f_2 variable. This instability corresponds to the bifurcation of the ANO vortices with the twisted vortex solutions [9]. The systematic expansion of a twisted vortex

near the bifurcation point can be then written as:

$$\begin{aligned}
 f_1 &= f_1^{(0)} + \epsilon^2 f_1^{(2)} + \dots \\
 f_2 &= \epsilon f_2^{(1)} + \epsilon^2 f_2^{(2)} + \dots \\
 a &= a^{(0)} + \epsilon^2 a^{(2)} + \dots \\
 a_3 &= \epsilon^2 a_3^{(2)} + \dots \\
 \omega &= \omega_b + \epsilon \omega_1 + \epsilon^2 \omega_2 + \dots
 \end{aligned} \tag{1.13}$$

where $a^{(0)}, f_1^{(0)}$ denotes the ANO vortex, whose equations can be read off from equations (1.9) by putting $f_2 = a_3 = 0$. In the above expansion, we have omitted some terms linear in ϵ . When solving the equations of the field components, these turn out to be zero (i.e. their equation is a homogeneous linear equation). As we shall see the only first order term is $f_2^{(1)}$. This is physically plausible too: the bifurcation is parameterized by the growing of the condensate f_2 .

The equations for the field components can be obtained easily by substituting the above expansion into the vortex profile equations (1.9). To first order in the ϵ expansion the only non-trivial equation determines the second component of the scalar field,

$$(D_2^{(0)} + \omega_b^2) f_2^{(1)} := -\frac{1}{r} (r f_2^{(1)'})' + \left[\frac{(n a^{(0)})^2}{r^2} - \beta (1 - (f_1^{(0)})^2) + \omega_b^2 \right] f_2^{(1)} = 0. \tag{1.14}$$

(The operator $D_2^{(0)}$ should not be confused with the $\mu = 2$ component of the covariant derivative D_μ .) In second order the equation for $f_2^{(2)}$ can be written as

$$(D_2^{(0)} + \omega_b^2) f_2^{(2)} = -2\omega_b \omega_1 f_2^{(1)}. \tag{1.15}$$

Eq. (1.15) contains only a resonance term on the right hand side, therefore the solution of this equation is $f_2^{(2)} = 0$ and the removal of the resonance term yields $\omega_1 = 0$.

$$\begin{aligned}
 & (r a_3^{(2)'})' / r - 2(f_1^{(0)})^2 a_3^{(2)} = -2(f_2^{(1)})^2, \\
 & (r f_1^{(2)'})' / r - \left[\frac{n^2(1-a^{(0)})^2}{r^2} - \beta(1-3(f_1^{(0)})^2) \right] f_1^{(2)} + f_1^{(0)} \frac{2n^2(1-a^{(0)})}{r^2} a^{(2)} = \beta f_1^{(0)} (f_2^{(1)})^2, \\
 & r \left(\frac{a^{(2)'}}{r} \right)' - 2(f_1^{(0)})^2 a^{(2)} - 4(a^{(0)} - 1) f_1^{(0)} f_1^{(2)} = 2a^{(0)} (f_2^{(1)})^2.
 \end{aligned} \tag{1.16}$$

The absence of the resonance term from the third order equations gives

$$\omega_2 = -\frac{1}{2\omega_b \|f_2^{(1)}\|} \left(f_2^{(1)}, \frac{2na^{(0)}}{r^2} n a^{(2)} f_2^{(1)} - 2\omega_b^2 a_3^{(2)} f_2^{(1)} + 2\beta f_1^{(0)} f_1^{(2)} f_2^{(1)} + \beta (f_2^{(1)})^3 \right), \tag{1.17}$$

where the scalar product and the norm is defined as

$$(f, g) := \int_0^\infty r dr f g, \quad \text{and} \quad \|f\|^2 := (f, f). \tag{1.18}$$

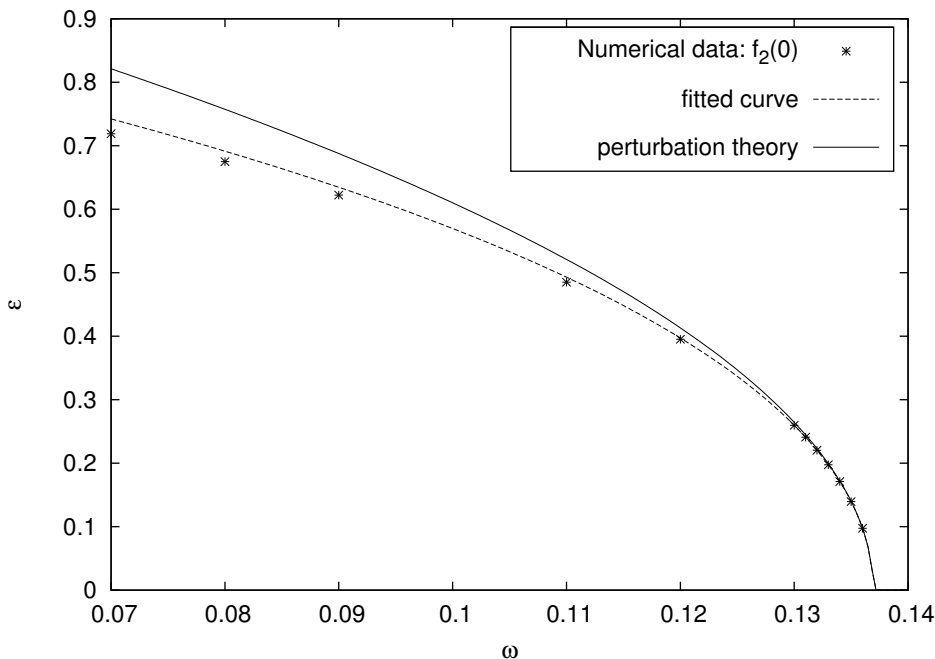


Figure 1. The ω - ϵ dependence, $\beta = 1.25$

Identifying $f_2(0)$ with the perturbation parameter, ϵ , at the bifurcation point, we obtain that

$$\epsilon = \sqrt{\frac{1}{\omega_2}(\omega - \omega_b)} + \dots \tag{1.19}$$

We have compared the results of the above perturbative bifurcation analysis, for $f_2(0)$ as a function of ω with numerical results on figure 1. A fitted curve, of the form $[(\omega - \omega_b)/\omega_2]^{1/2} + \epsilon_1(\omega_b - \omega)$, with ω_2 and ϵ_1 being the parameters fitted, as suggested by perturbation theory, is also shown. See also table 5. Another nice comparison of the twisted and ANO vortices can be seen on figures 2–3. We have drawn no figures of f_1 and a . Their perturbation series starts with a zeroth order ANO term, and the next correction is of order ϵ^2 , therefore, the difference between the ANO background, the second order perturbative result and the exact twisted vortex profile function is quite small.

As we can see from both the comparison of the ϵ - ω dependence and the vortex profile functions, the above perturbative method gives a good approximation of twisted vortices, even relatively far from the bifurcation, i.e. for $\epsilon = 0.2 \dots 0.3$. In subsections 2.2 and 2.3 we will use this perturbatively obtained vortex background to explore the relation between the instabilities of the embedded ANO vortices and those of the twisted ones.

2 Linear stability analysis

2.1 Perturbation analysis

Here we present a stability analysis of the twisted vortex solution based on the linearization of the field equations about the vortex. Let $\phi_a = \phi_{a,bg} + \epsilon\delta\phi_a$ and $A_\mu = A_{\mu,bg} + \epsilon\delta A_\mu$, with

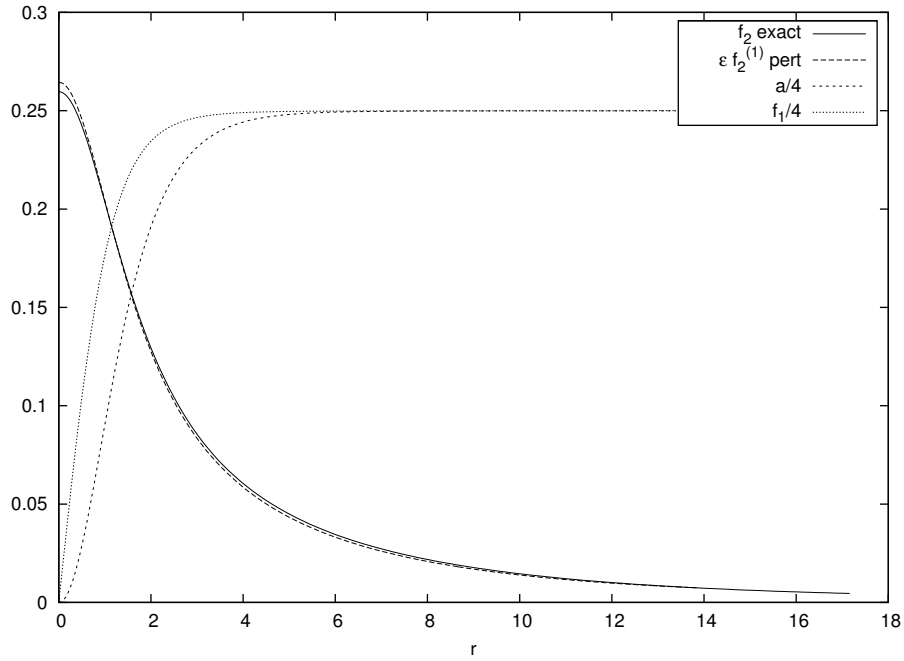


Figure 2. The twisted vortex profile function f_2 for $\beta = 1.25$, $\omega = 0.13$. For comparison, we have also plotted f_1 and a in this figure (note the scaling by $1/4$ of f_1 and a).

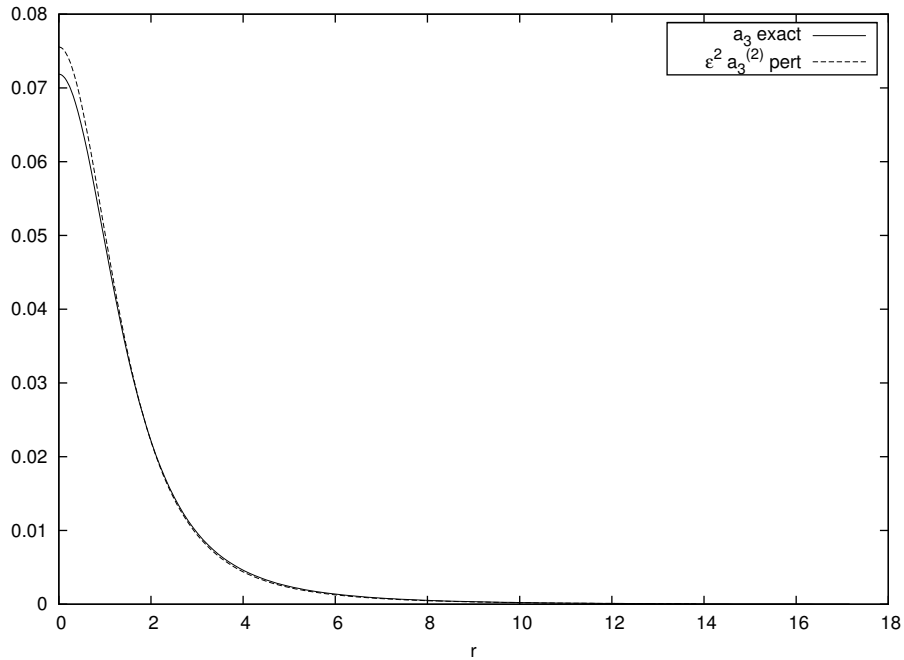


Figure 3. The twisted vortex profile function a_3 for $\beta = 1.25$, $\omega = 0.13$

$\phi_{a,\text{bg}}$ and $A_{\mu,\text{bg}}$ being the background solution. The index “bg” will be dropped in what follows; the total perturbed fields will not appear later on. Here, we will summarize the main properties and the method of solution of the linear equations governing $\delta\phi_a$ and δA_μ . Fol-

lowing the footsteps of ref. [11] we will use the background field gauge. The main differences between the present case and that of the ANO vortices is that here there are more field components and the z -dependence does not decouple, which makes the problem computationally more demanding. The main advantage of the background field gauge is the simplicity of the resulting equations. Its main drawback is, that, unlike the method used by ref. [16], it does not fix the gauge completely, which makes an analysis of ghost modes necessary.

The perturbation equations can be derived from a linearization of the field equations. The resulting equations, taking into account that the background solution satisfies the Lorentz gauge condition $\partial^\mu A_\mu = 0$, can be cast into the form

$$D \begin{pmatrix} \delta\phi_a \\ \delta\phi_a^* \\ \delta A^\mu \end{pmatrix} = 0, \quad (2.1)$$

where the operator D is

$$\begin{pmatrix} D_1 + 2i\delta_{ab}A^\mu\partial_\mu - \beta\phi_b\phi_a^* & -\beta\phi_b\phi_a & 2i\partial_\mu\phi_b + 2A_\mu\phi_b + i\phi_b\partial_\mu \\ -\beta\phi_b^*\phi_a^* & D_1 - 2i\delta_{ab}A^\mu\partial_\mu - \beta\phi_b^*\phi_a & -2i\partial_\mu\phi_b^* + 2A_\mu\phi_b^* - i\phi_b^*\partial_\mu \\ -i\partial_\nu\phi_a^* + 2A_\nu\phi_a^* + i\phi_a^*\partial_\nu & i\partial_\nu\phi_a + 2A_\nu\phi_a - i\phi_a\partial_\nu & |\phi|^2 2g_{\mu\nu} - \partial_\mu\partial_\nu + g_{\mu\nu}\square \end{pmatrix}, \quad (2.2)$$

with $D_1 = -\delta_{ab}\square + A_\mu A^\mu \delta_{ab} - \beta(|\phi|^2 - 1)\delta_{ab}$ and $\square = \partial_\mu\partial^\mu$.

In the $\beta \rightarrow \infty$ limit, similarly to the regular case, the Lagrangian is expanded to second order in $\delta\phi_a$, $\delta\phi_a^*$ and δA_μ . Denoting the zeroth and first order terms of $D_\mu D^\mu \phi_a$ with χ_a and ψ_a , equation (A.8) can be written as $\psi_a = 0$. Now, the linearization of the Lagrange multiplier (1.7) has to be added. The resulting equations of motion are

$$\begin{aligned} \psi_a - (\lambda^{(0)}\delta\phi_a + \lambda^{(1)}\phi_a) &= 0, \\ \psi_a^* - (\lambda^{(0)}\delta\phi_a^* + \lambda^{(1)}\phi_a^*) &= 0, \end{aligned} \quad (2.3)$$

where

$$\begin{aligned} \lambda^{(0)} &= \frac{1}{2}(\phi_a^*\chi_a + \phi_a\chi_a^*), \\ \lambda^{(1)} &= \frac{1}{2}(\delta\phi_a^*\chi_a + \phi_a^*\psi_a + \delta\phi_a\chi_a^* + \phi_a\psi_a^*). \end{aligned} \quad (2.4)$$

The perturbation equations are invariant under infinitesimal gauge transformations of the form

$$\begin{aligned} \delta\phi_a &\rightarrow \delta\phi_a + i\chi\phi_a, \\ \delta A_\mu &\rightarrow \delta A_\mu + \partial_\mu\chi. \end{aligned} \quad (2.5)$$

This gauge freedom can be dealt with by adding a gauge-fixing term $\frac{1}{2}|F(A)|^2$ to the Lagrangian,

$$F(A) := \partial_\mu\delta A^\mu + i(\delta\phi_a^*\phi_a - \phi_a^*\delta\phi_a) = 0. \quad (2.6)$$

This helps one to get rid of the terms containing first order derivatives in D , yielding the perturbation matrix

$$\begin{pmatrix} D_1 + 2i\delta_{ab}A^\mu\partial_\mu - (\beta+1)\phi_b\phi_a^* & -(\beta-1)\phi_b\phi_a & 2i\partial_\mu\phi_b + 2A_\mu\phi_b \\ -(\beta-1)\phi_b^*\phi_a^* & D_1 - 2i\delta_{ab}A^\mu\partial_\mu - (\beta+1)\phi_b^*\phi_a & -2i\partial_\mu\phi_b^* + 2A_\mu\phi_b^* \\ -2i\partial_\nu\phi_a^* + 2A_\nu\phi_a^* & 2i\partial_\nu\phi_a + 2A_\nu\phi_a & |\phi|^2 2g_{\mu\nu} + g_{\mu\nu}\square \end{pmatrix}, \quad (2.7)$$

where $D_1 = -\delta_{ab}\square + A_\mu A^\mu \delta_{ab} - \beta(|\phi|^2 - 1)\delta_{ab}$.

It can be easily verified that the equation of δA_0 ,

$$\partial_\mu \partial^\mu \delta A_0 + 2|\phi|^2 \delta A_0 = 0, \quad (2.8)$$

decouples, just as in the case of the ANO vortex [11].

The gauge condition (2.6) does not fix the gauge completely, but still allows gauge transformations whose generating function satisfies the ghost mode equation

$$\partial_\mu \partial^\mu \chi + 2|\phi|^2 \chi = 0. \quad (2.9)$$

The ghost modes cancel the δA_0 modes, whose eq. (2.8) is identical to (2.9), and a part of the spectrum of the gauge fixed operator (2.7). In the case of the embedded ANO vortices, a straightforward analysis of ghost modes is possible, see appendix B.

Let us now apply a Fourier transform to the perturbation equations (2.7) in the t and z variables. Introducing $\Psi = (\delta\phi_1, \delta\phi_1^*, \delta\phi_2, \delta\phi_2^*, \delta A_i, \delta A_3, \delta A_0)^T$.

$$\Psi = \int dk d\Omega e^{i(\Omega t - kz)} \tilde{\Psi}. \quad (2.10)$$

This way, the perturbation equations can be brought into the form

$$\mathcal{M}\tilde{\Psi} = \Omega^2 \tilde{\Psi}. \quad (2.11)$$

See appendix A for the details of the calculation and the matrix \mathcal{M} . The angle variable ϑ can be separated in a similar fashion, by the Fourier series expansion

$$\tilde{\Psi} = \sum_\ell \Psi_\ell e^{i\ell\vartheta}. \quad (2.12)$$

This yields the desired eigenvalue problem

$$M_\ell \Psi_\ell = \Omega^2 \Psi_\ell, \quad (2.13)$$

where M_ℓ is an ordinary differential operator in the radial variable r (see equation (A.8) in appendix A). A similar expansion can be done for the ghost modes.

The numerical method used for solving equations (2.13) as well as the background equations (1.9) was shooting to a fitting point. The resulting eigenvalues as a function of k can be seen on figure 4 for $\beta = 1.25, 2, 2.5$ and the limit $\beta \rightarrow \infty$. There are two unstable modes, one which is the deformation of the unstable mode in s_2 of the embedded ANO vortex, which has a negative eigenvalue for $0 < k < k_m$, and another one, which is the deformation of the unstable mode in s_2^* of the embedded ANO vortex and has a negative eigenvalue for $0 > k > -k_m$. These two modes are related by a transformation of $k \rightarrow -k$ and interchanging the fields s_1, s_2, a_+ and the conjugate fields s_1^*, s_2^*, a_- . The numerical results exhibit this symmetry to a rather high accuracy.

In figure 4, we have plotted the dispersion relation of the unstable modes for $0 < k < k_m$. A closed form of this is not known, however, the approximation

$$\Omega^2 = \Omega_{\min}^2 + \Omega_2^2 (k - k_{\min})^2 \quad (2.14)$$

ω	Ω_{\min}^2	Ω_2^2	k_{\min}
0.07	-0.00105	0.73927	0.05418
0.08	-0.00173	0.80044	0.06383
0.09	-0.00262	0.84997	0.07352
0.11	-0.00591	0.94158	0.09542
0.12	-0.00906	0.97588	0.10882
0.13	-0.01390	0.99548	0.12435
0.136	-0.01801	0.99984	0.13513

Table 1. Twisted vortex perturbations: parameters of the dispersion relation for $\beta = 1.25$

ω	Ω_{\min}^2	Ω_2^2	k_{\min}
0.13	-0.00197	0.53731	0.09140
0.15	-0.00293	0.57959	0.10545
0.2	-0.00812	0.73531	0.14689
0.25	-0.02213	0.88462	0.19989
0.3	-0.05964	0.97901	0.27098
0.32	-0.08905	0.99561	0.30837
0.329	-0.10687	0.99969	0.32785

Table 2. Twisted vortex perturbations: parameters of the dispersion relation for $\beta = 2$

is empirically found to be very good. Note that k_{\min} , the wavenumber corresponding to the minimal eigenvalue, should not be confused with k_m standing for the maximal value of the wave numbers of the unstable modes. In figure 5 one can see Ω_{\min}^2 and k_{\min} as a function of ω for $\beta = 1.25$. The same data, for $\beta = 1.25, 2$ and 2.5 is summarized in tables 1, 2 and 3. In the case of the embedded ANO vortex, the form (2.14) of the dispersion relation is exact. In ref. [16] it is claimed that the minimum value is achieved at $k = \pm\omega$, but for smaller values of ω we have found a considerable deviation from $k_{\min} = \pm\omega$ (see especially the curves corresponding to $\beta \rightarrow \infty$).

Close to the bifurcation, our numerical results agree with those of ref. [16]. However, for smaller values of ω , there is some discrepancy which we attribute to numerical errors.

It is important to note, that the modes discussed in this section are all part of the physical spectrum. This can be verified by solving the ghost equation (2.9). The ghost modes we obtained do not have nodes and correspond to a positive eigenvalue, therefore there cannot be lower eigenvalue modes. This shows, that the instability modes presented in this paper are all physical (i.e. there are no negative eigenvalue ghost modes which could cancel them).

For the $\beta \rightarrow \infty$ case the results of the numerical calculations can be seen (with the parametrization used in the $\beta < \infty$ case) in table 4. See also the dispersion relation in figure 4. In this case, there are no ghost mode solutions (note also that in the regular case, the ghost mode eigenvalues increase for larger values of β). The $\beta \rightarrow \infty$ data shows that the instability modes persist for all values of β . For $\beta \rightarrow \infty$ $\omega_b \rightarrow \infty$, thus the $\beta \rightarrow \infty$

ω	Ω_{\min}^2	Ω_2^2	k_{\min}
0.14	-0.00211	0.50713	0.09848
0.15	-0.00249	0.51283	0.10565
0.2	-0.00498	0.57028	0.13811
0.25	-0.01112	0.69774	0.17859
0.3	-0.02468	0.82209	0.22846
0.35	-0.05385	0.92445	0.28949
0.4	-0.11798	0.98491	0.36985
0.42	-0.16215	0.99685	0.41056
0.427	-0.18142	0.99984	0.42641

Table 3. Twisted vortex perturbations: parameters of the dispersion relation for $\beta = 2.5$

ω	Ω_{\min}^2	Ω_2^2	k_{\min}
0.1	-0.000155	0.284781	0.042232
0.5	-0.007838	0.298390	0.266817
1	-0.053988	0.364306	0.585068
2	-0.402749	0.499033	1.356712
3	-1.195500	0.571684	2.179592
4	-2.466004	0.618034	3.016282
5	-4.215287	0.643379	3.855931

Table 4. Twisted vortex perturbations: parameters of the dispersion relation for $\beta \rightarrow \infty$

β	ω_b	ω_2	ω_b^{fit}	ω_2^{fit}	$(\Omega^2)^{(0)}$	$(\Omega^2)^{(2)}$	$(\Omega^2)_{\text{fit}}^{(0)}$	$(\Omega^2)_{\text{fit}}^{(2)}$	α
1.25	0.13694	-0.09926	0.13694	-0.09790	-0.0188	0.111	-0.0188	0.0802	—
2.0	0.32992	-0.17824	0.32989	-0.17815	-0.1088	0.488	-0.1088	0.3914	0.614
2.5	0.42744	-0.21244	0.42744	-0.21225	-0.1827	0.720	-0.1826	0.6003	0.501

Table 5. Perturbative and numerical data of vortex instability modes. Here, the subscript “fit” denotes data obtained by fitting a parabola to the numerically obtained dispersion relation, close to the minimum.

data can be used to study vortices with values of $\omega \ll \omega_b$. Our data indicates that for β fixed, the family of unstable modes parameterized by $k \in [-k_m, k_m]$ also persists for all values of $0 < \omega < \omega_b$.

2.2 Bifurcation analysis of the perturbation operator

For values of the twist $\omega \approx \omega_b$ we have expanded the vortex backgrounds in a suitable parameter ϵ . In this paragraph we give the second order expansion of the perturbation operator and its eigenvalues in the bifurcation parameter ϵ . For details, see appendix A.3.

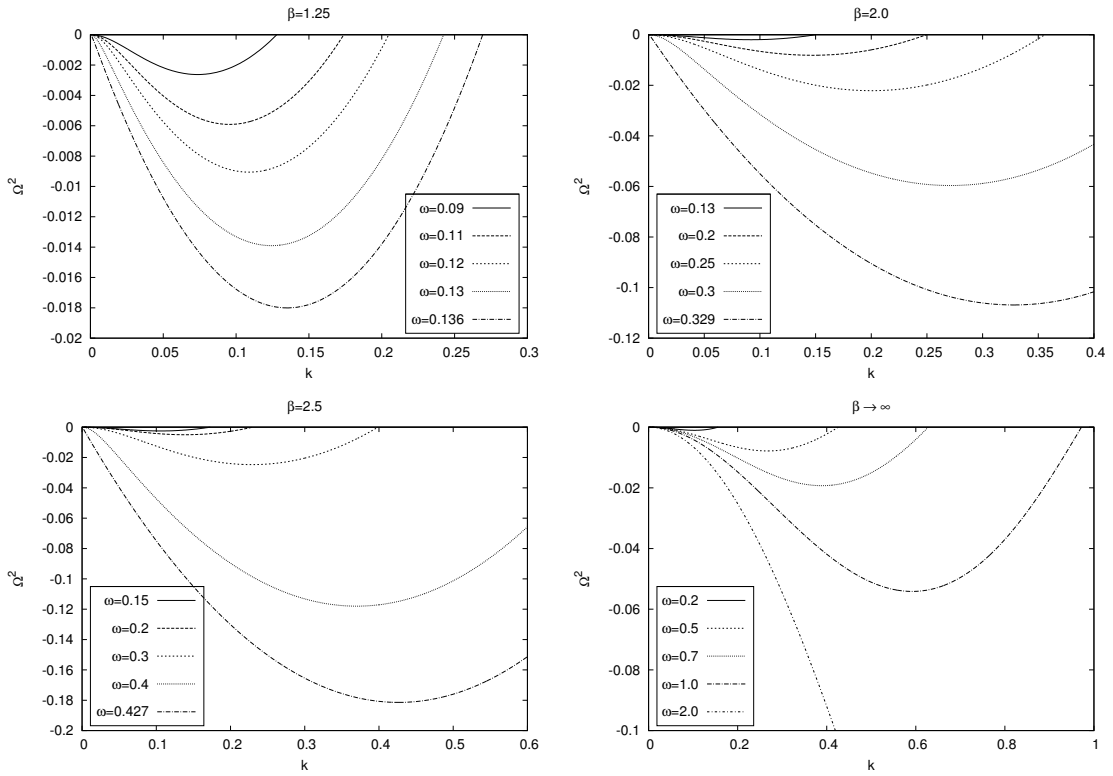


Figure 4. The dispersion relation of the unstable mode

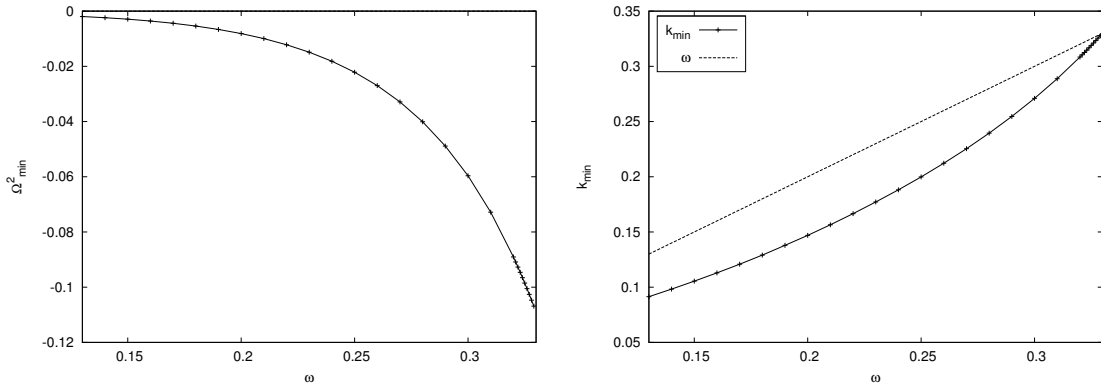


Figure 5. Minima of Ω^2 and its location for $\beta = 2$ as a function of ω

The leading order equation of the mode pertinent to our problem is

$$D_2^{(0)} s_2 = (\Omega^2)^{(0)} s_2, \tag{2.15}$$

which yields the unstable mode of the embedded ANO vortex. Let us note here, that for $k = \omega$, this instability mode is z -independent. For other values of k , its z -dependence is harmonic.

Let us now look at the first order correction to the eigenvalue. This correction, as

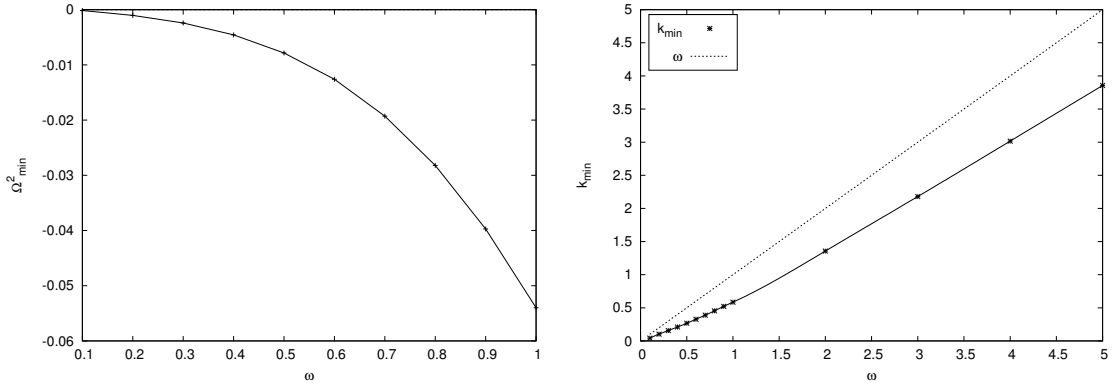


Figure 6. Minima of Ω^2 and its location for $\beta \rightarrow \infty$ as a function of ω

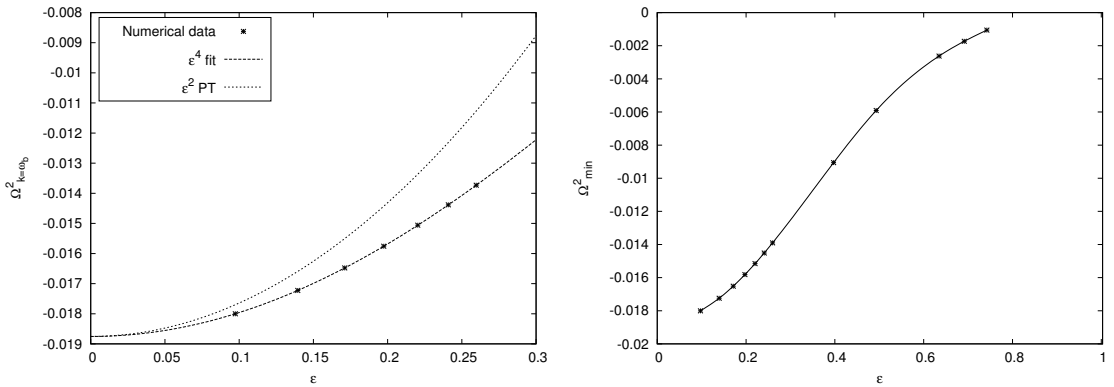


Figure 7. $\Omega_{k=\omega_b}^2$ and its comparison with PT and the dependence of the minimal Ω^2 on ϵ for $\beta = 1.25$

known from quantum mechanics is given as

$$(\Omega^2)^{(1)} = \frac{1}{\|s_2\|^2} (s_2, D_2^{(1)} s_2) = 0. \tag{2.16}$$

Therefore, to obtain an ω -dependent eigenvalue, one must go to the second order in perturbation theory. Contributions quadratic in ϵ come from two sources: from $\mathcal{O}(\epsilon^2)$ terms in the perturbation operator (i.e. corrections to the background functions of order ϵ^2), and from second order perturbation theory,

$$(\Omega^2)^{(2)} = (\Omega^2)_B^{(2)} + (\Omega^2)_P^{(2)}. \tag{2.17}$$

The first term is simply

$$(\Omega^2)_B^{(2)} = \frac{1}{\|s_2\|^2} (s_2, D_2^{(2)} s_2) = M_{22} + M_{22}^k (k - \omega_b) \tag{2.18}$$

with

$$M_{22} = \frac{1}{\|s_2\|^2} \left(s_2, \left[-\frac{2(\ell - na^{(0)})na^{(2)}}{r^2} + 2\beta f_1^{(0)} f_1^{(2)} + (2\beta + 1)(f_2^{(1)})^2 \right] s_2 \right) \tag{2.19}$$

and

$$M_{22}^k = \frac{1}{\|s_2\|^2} \left(s_2, \left[-2\omega_2 + 2\omega_b a_3^{(2)} \right] s_2 \right). \quad (2.20)$$

To calculate the other contributions, let us first introduce the following notations (δA_0 component dropped — it is decoupled in all orders; see also appendix B.)

$$\begin{aligned} \psi_2 &:= (0, 0, s_2, 0, 0, 0)^T, \\ \psi_2^\dagger &:= (0, 0, 0, s_2^*, 0, 0)^T, \\ \psi_S &:= (s_0^S, s_0^S, 0, 0, a_0^S, a_0^S)^T, \\ \psi_A &:= (s_0^A, -s_0^A, 0, 0, a_0^A, -a_0^A)^T, \\ a_3 &:= (0, 0, 0, 0, 0, a_3)^T. \end{aligned} \quad (2.21)$$

In eq. (2.21), ψ_2 is the unstable mode of the embedded ANO vortex (see eq. (2.15)) and ψ_2^\dagger is its conjugate mode (it can be seen easily, that the equation of s_2^* is the same as that of s_2 with the transformation $k \rightarrow -k$). ψ_S and ψ_A are the symmetric and antisymmetric bound modes of the ANO vortex, respectively, while a_3 is another bound mode of the ANO vortex, satisfying the same equation as ghosts.

The dispersion relation of these modes (in the same order as in eq. (2.21)) is given as

$$\begin{aligned} \Omega_{s_2}^2 &= \lambda_2^2 + (k - \omega)^2, \\ \Omega_{s_2^*}^2 &= \lambda_2^2 + (k + \omega)^2, \\ \Omega_S^2 &= \lambda_S^2 + k^2, \\ \Omega_A^2 &= \lambda_A^2 + k^2, \\ \Omega_{a_3}^2 &= \lambda_{a_3}^2 + k^2, \end{aligned} \quad (2.22)$$

with λ_i being constants (lowest eigenvalues). For second order perturbation theory, the following matrix elements are needed,

$$\begin{aligned} M_{22^*} &= \frac{1}{\|s_2\|^2} \left(s_2, U_2^{(1)} s_2^* \right) = 0, \\ M_{2S} &= \frac{1}{\|s_2\| \|\psi_S\|} \left[\left(s_2, (V^{(1)} + V'^{(1)}) s_0^S \right) + \left(s_2, (A_2^{(1)} + A_2'^{(1)}) a_0^S \right) \right], \\ M_{2A} &= \frac{1}{\|s_2\| \|\psi_A\|} \left[\left(s_2, (V^{(1)} - V'^{(1)}) s_0^A \right) + \left(s_2, (A_2^{(1)} - A_2'^{(1)}) a_0^A \right) \right], \\ M_{23} &= \frac{1}{\|s_2\| \|a_3\|} \left(s_2, B_2^{(1)} a_3 \right). \end{aligned} \quad (2.23)$$

The part of the $\mathcal{O}(\epsilon^2)$ correction to the eigenvalue due to perturbation theory can be written as

$$(\Omega^2)_P^{(2)} = \frac{|M_{2S}|^2}{\lambda_S^2 - 2\omega_b^2 + 2k\omega_b} + \frac{|M_{2A}|^2}{\lambda_A^2 - 2\omega_b^2 + 2k\omega_b} + \frac{|M_{23}|^2}{\lambda_{a_3}^2 - 2\omega_b^2 + 2k\omega_b} \quad (2.24)$$

The contribution to the perturbation functions will be evaluated similarly using the formulas well known from elementary quantum mechanics in subsection 2.3.

Looking at the potentials in the equations of ψ_S and ψ_A (see figure 10 in appendix B), one can easily understand why the mode ψ_S becomes quasi-bound (instead of bound) for values of β above 1.5 (see appendix B). The contribution of these modes to PT has to be taken into account when calculating the perturbation (2.24) of the eigenvalue. This can be done as follows: the s -component of the quasi-bound mode varies slowly with the energy in a given interval, while the a -component varies rapidly. This makes it possible, to smoothen the contribution by using a low-lying quasi-bound mode, with the a -component taken to be zero and reading off the integral of the state density over the energy interval, in which the s -component is nearly constant, from the perturbation functions, and using a correction factor α to calculate the energy correction:

$$(\Omega^2)_P^{(2)} = \frac{\alpha|M_{2S}|^2}{\lambda_S^2 - 2\omega_b^2 + 2k\omega_b} + \frac{|M_{2A}|^2}{\lambda_A^2 - 2\omega_b^2 + 2k\omega_b} + \frac{|M_{23}|^2}{\lambda_{a3}^2 - 2\omega_b^2 + 2k\omega_b} \quad (2.25)$$

with the matrix element M_{2S} calculated from the smoothed ψ_S , and α is obtained from a comparison of the perturbative and exact eigenmodes.

The eigenvalue Ω^2 at $k = \omega_b$ as a function of ϵ (near $\omega = \omega_b$) can easily be calculated with perturbation theory as outlined in the previous section. Second order perturbation theory predicts

$$\Omega_{k=\omega_b}^2 = (\Omega^2)^{(0)} + \epsilon^2(\Omega^2)^{(2)}. \quad (2.26)$$

Values of $(\Omega^2)^{(0)}$ and $(\Omega^2)^{(2)}$ obtained with the perturbation theory methods of the previous section and exact (numerical) values are compared in table 5 and figure 7. Similarly to the case of the vortex backgrounds close to the bifurcation, there is a good agreement between the exact eigenvalues and the perturbative ones, for moderate values of ϵ , i.e. approximately for $\epsilon \leq 0.2$. This is an indication that the instability mode of the twisted vortex can be regarded as a deformation of the instability mode of the embedded ANO vortex.

2.3 Perturbative calculation of the eigenfunction corresponding to the instability

In this section, we present the perturbation functions $s_{1,\ell}, s_{1,-\ell}^*, s_{2,\ell}, s_{2,-\ell}^*, a_\ell, a_{-\ell}^*, a_{3,\ell}$. We will also compare these with their first order perturbative expressions near the bifurcation point.

Let us first note, that for vortices close to the bifurcation (i.e. ω close to ω_b , $\epsilon \ll 1$), the lowest lying negative eigenvalue is close to $k = \omega$. Let us now choose a vortex which can still be treated perturbatively, but it is not very close to an embedded ANO vortex, eg. $\omega = 0.13$ for $\beta = 1.25$, and examine its instability mode for $k = \omega$.

In the previous subsection, we have applied perturbation theory to obtain the eigenvalues of the perturbation operator. Let us now apply PT to evaluate the wave function, i.e. the unstable mode of the twisted vortex. The unperturbed wave function is simply the instability mode of the embedded ANO vortex,

$$\psi^{(0)} = \psi_2 = (0, 0, s_2, 0, 0, 0, 0)^T. \quad (2.27)$$

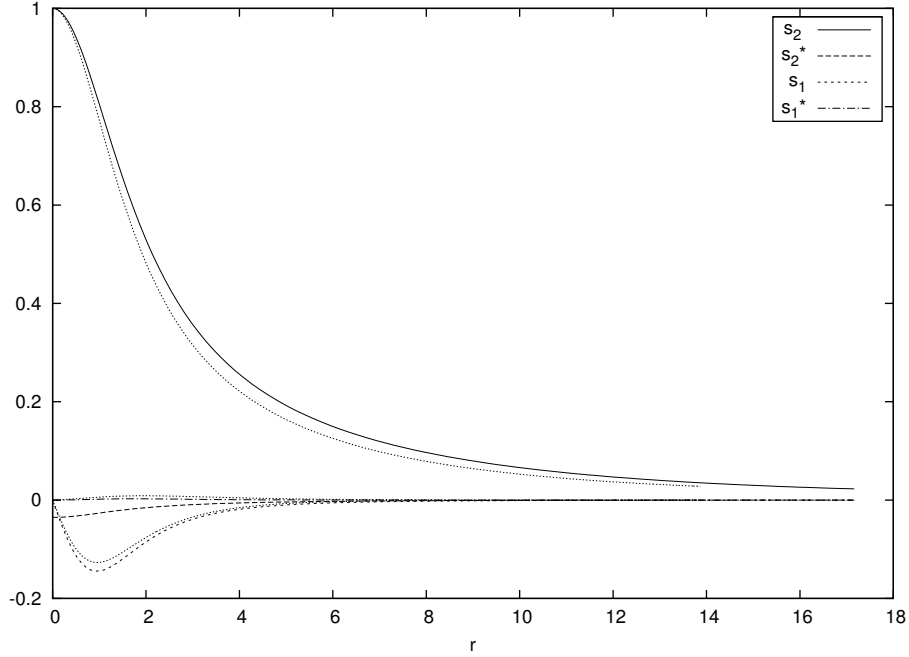


Figure 8. The twisted vortex instability mode, scalar field components for $\beta = 1.25$, $\omega = 0.13$. Perturbative results for s_1 , s_1^* and s_2 are shown with dotted lines, $s_2^* \equiv 0$ to this order.

Let us now write up the first order perturbations to this mode,

$$\psi^{(1)} = -N \frac{M_{2S}}{\lambda_S^2 - 2\omega_b^2 + 2k\omega_b} \psi_S - N \frac{M_{2A}}{\lambda_A^2 - 2\omega_b^2 + 2k\omega_b} \psi_A - N \frac{M_{23}}{\lambda_3^2 - 2\omega_b^2 + 2k\omega_b} a_3 \quad (2.28)$$

where $N = \|\psi_2\|$. The first order wavefunction and the exact instability mode exhibit good agreement, even for not so small values of ϵ (see figures 8 and 9; in that case $\epsilon = 0.264$).

Note that the perturbations depicted in figures 8–9, correspond to a quite large value of ϵ , and even so the instability mode components are still strongly dominated by s_2 . This indicates that the instability of the twisted vortex proceeds similarly to that of the embedded ANO one, i.e. the vortex starts to expand nearly homogeneously. To support this conclusion further, we calculate the perturbations of the current of the vortex. The components, which receive perturbations, are

$$\delta j_0^3 = e^{i\ell\vartheta} e^{i(\Omega t - kz)} \Omega \{ f_1(s_{1,-\ell}^* - s_{1,\ell}) - f_2(s_{2,-\ell}^* - s_{2,\ell}) \}, \quad (2.29)$$

where, $s_{1,-\ell}^* = \mathcal{O}(\epsilon)$, $s_{1,\ell} = \mathcal{O}(\epsilon)$, and $f_2 = \mathcal{O}(\epsilon)$, i.e. this perturbation does not change the zero component of the current in the leading order, and

$$\begin{aligned} \delta j_3^3 = e^{i\ell\vartheta} e^{i(\Omega t - kz)} \{ & (k + 2\omega a_3) f_1 s_{1,\ell} - (k + 2\omega(a_3 - 1)) f_2 s_{2,\ell} \\ & + (-k + 2\omega a_3) f_1 s_{1,-\ell}^* - (-k + 2\omega(a_3 - 1)) f_2 s_{2,-\ell}^* + 2(f_1^2 - f_2^2) a_{3,\ell} \}, \end{aligned} \quad (2.30)$$

where $s_{1,-\ell}^* = \mathcal{O}(\epsilon)$, $s_{1,\ell} = \mathcal{O}(\epsilon)$, $a_{3,\ell} = \mathcal{O}(\epsilon)$ and $f_2 = \mathcal{O}(\epsilon)$ therefore $\delta j_3^3 = \mathcal{O}(\epsilon)$ too.

The deviation of both the eigenvalues and the eigenfunctions of twisted strings is of $\mathcal{O}(\epsilon^2)$ with respect to those of the ANO ones. This suggests that the eigenmode of the

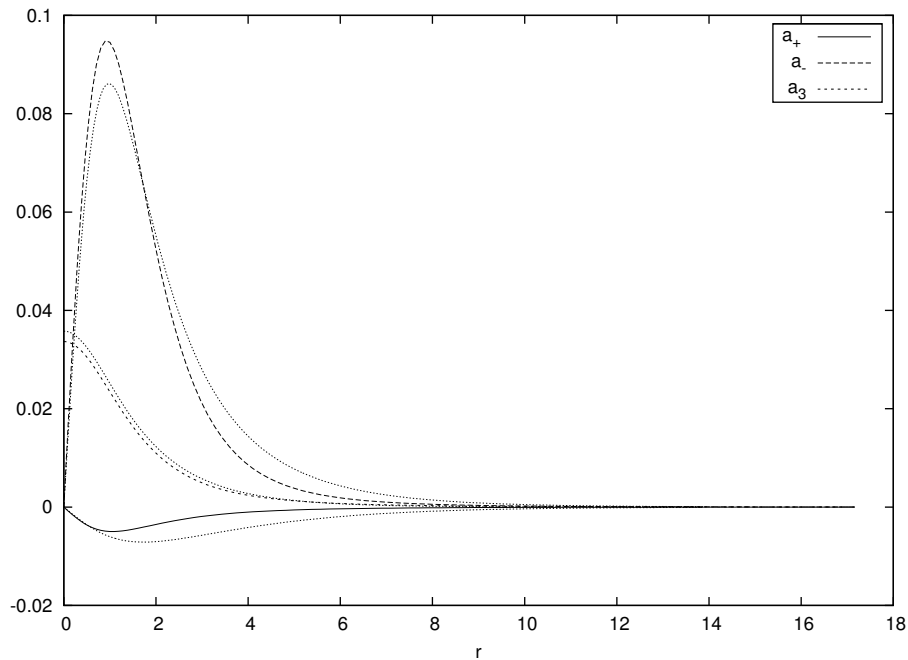


Figure 9. The twisted vortex instability mode, vector field components for $\beta = 1.25$, $\omega = 0.13$. Perturbative results are shown with dotted lines.

β	M_{22}	M_{22}^k	$(\Omega^2)^{(0)}$	ϵ_{\max}
1.25	0.6883	0.2934	-0.0188	0.1653
2	1.2442	0.5990	-0.1088	0.2957
2.5	1.4843	0.7270	-0.1827	0.3396

Table 6. Data for the calculation of the energy of the mode ψ_2 on a twisted vortex background

ANO string is an energy lowering perturbation of the twisted one. The energy, $\Omega^2 = (\psi_2, D_2 \psi_2)$, of the embedded ANO instability mode, s_2 , on a twisted vortex background can be expanded as

$$\Omega^2 = (\Omega^2)^{(0)} + \epsilon^2 (\Omega^2)^{(2)}. \tag{2.31}$$

Using the expansion of the perturbation operator

$$D_2 = D_2^{(0)} + \epsilon^2 D_2^{(2)}, \tag{2.32}$$

yields

$$(\Omega^2)^{(2)} = M_{22} + M_{22}^k (k - \omega_b). \tag{2.33}$$

Table 6 shows clearly the important fact that for not too large values of ϵ , the perturbation s_2 still lowers the energy. The ϵ_{\max} values were calculated with $k = \omega_b$.

2.4 Properties of the instability modes

Let us recapitulate first, that, as shown in subsection 2.1, the obtained unstable eigenmodes are all physical. An examination of the above calculated eigenvalues and the corresponding

eigenfunctions shows clearly that the obtained instability eigenmode is a deformation of the instability eigenmode of the embedded ANO vortices already mentioned in section 1.3 (See also appendix B), in the sense that for a given value of the z -direction wavenumber k , the corresponding eigenmode Ψ and eigenvalue Ω are smooth functions of the twist ω , which, for $\omega = \omega_b$ reproduce the eigenmodes and eigenvalues of the embedded ANO vortices. In subsection 2.3 we presented a description of this deformation based on perturbation theory, with the unperturbed solutions being the eigenmodes of the embedded ANO vortex.

The most important result is that for each $k \in [-k_m, k_m]$ there is one eigenmode corresponding to a negative eigenvalue (here k_m , of course, depends on ω). In this section we would like to summarize the properties of these eigenmodes, and speculate about the physical consequences of them.

Firstly, let us briefly analyze the properties of the instability modes. Let us point out that even for values of ω quite far from the bifurcation ω_b ($\epsilon \approx 0.2 \dots 0.3$), the unstable mode is dominated by the s_2 component (see also section 2.3 and figures 8–9 therein), and the lowest lying eigenvalue is close to $k = \omega$. If $k = \omega$ then the dominant s_2 component of the unstable mode is z -independent. In the previous subsection we have shown, that for $\omega \approx \omega_b$ (see table 6 for critical values of ϵ at which the energy of this perturbation becomes positive) the unstable mode of the ANO vortex is also an energy lowering perturbation for the twisted one, albeit not an eigenmode. In this way we can explicitly construct a z -independent energy-lowering perturbation of the twisted string, which is an indication that homogeneous expansion of twisted strings close to the bifurcation is possible.

The deviation from $k_{\min} = \omega$ occurs at small values of ω , far from the bifurcation, $\omega = \omega_b$, i.e. the z -dependence of the instability eigenmode becomes important when the vortices have already quite expanded in the lower scalar field component. In our opinion, this is another indication that the instability corresponds to an expansion of the string.

In order to see how the string can expand in case a completely homogeneous expansion is not possible, let us take a perturbation of the form of a wave packet, centered around $z = 0$ in coordinate space and around $k = k_{\min}$ in momentum space. Then, looking at the Fourier transform of such a wave packet one gets $\Delta x \Delta k = 1$ for the width of the wave packet in coordinate and momentum space, respectively. Let the amplitude of the packet be A . Then, the energy of this packet can be calculated as

$$E = |A|^2 [\Omega_{\min}^2 + \Omega_2^2(\Delta k)^2] + \dots,$$

which can be negative if Δk is sufficiently small. However, for the wave packet to make the current grow (locally make the vortex resemble a vortex of lower ω , i.e. let the vortex locally get diluted), one needs

$$\Delta x < \lambda = \frac{2\pi}{k_{\min}},$$

which gives

$$\Delta k \approx \frac{k_{\min}}{2\pi}.$$

Substituting this into the expression for the energy, at least close to the bifurcation point, one can see that

$$E \approx |A|^2 \left(\frac{\Omega_2^2}{4\pi^2} - 1 \right) k_{\min}^2 < 0.$$

We have also checked, using the values of Ω_{\min}^2 and Ω_2^2 obtained above, that for $\beta = 2$ for all values of ω available for our numerical methods the energy contribution of the lump can be negative, which suggests that the instability of the vortex corresponds to an expansion instability, similarly to the case of the embedded ANO vortices. This result makes it plausible that a local (lump-like) spreading of the vortex resulting in a configuration with diluted magnetic flux is possible. Furthermore, a (local) expansion of the vortex can form out of a general initial perturbation of the string, because the eigenmodes which are almost z -independent are the ones which are close to the mode corresponding to the largest negative eigenvalue, and therefore they are the ones that grow most rapidly in time, and a general initial perturbation (like the lump discussed above) overlaps with these modes.

The existence of the z -independent energy-lowering perturbations for strings close to the bifurcation and local lumps for all strings independently of ω clearly show that at least at the outset of the instability an expansion of the string is possible. In the case of embedded ANO vortices it is known [3, 10], that they are unstable against expansion. The similarity of the eigenmodes of twisted and embedded ANO strings supports that at least initially, twisted strings also start to expand. However, in the linear approximation, the perturbations of the current components, eq. (2.29) and (2.30), are harmonic in z , and thus the conserved current of the twisted string remains localized. This changes the dynamics of the expansion, compared to that of the embedded ANO strings considerably; the expansion cannot go on indefinitely until the current is carried away by radiation. Radiation effects only occur in higher orders of perturbation theory, and thus the framework of the present paper, i.e. linearization, is not sufficient to draw definite conclusions about the long term dynamics of the instability, and the resulting final state.

The stability of the global version of current carrying strings, in a theory with two scalar fields, has been analyzed in ref. [25]. It has been found that the string is stable against homogeneous and only z -dependent perturbations, and above a critical value of the (global) current unstable against time dependent ones. For twisted vortices in the semilocal model, it is the magnetic flux in the string that is responsible for the spreading instability.

Ref. [16] suggests another scenario for the instability of semilocal twisted strings, based on the harmonic z -dependence of the SU(2)-current flowing in the string (see eqs. (2.29) and (2.30) in the previous subsection), and the fact that a larger current corresponds to a more expanded (lower ω) string. There, an analogy with the Plateau-Rayleigh instability of a fluid stream (droplet formation) is emphasized. The analogue of the Plateau-Rayleigh instability, as suggested by ref. [16] is the break-up of the string into small, droplet-like string segments.

As mentioned above, there are good reasons to believe that for not too large values of the current, twisted strings start to expand initially. For large values of the current where the nearly homogeneous expansion is not an energy lowering perturbation, the width of

the string is already quite large, therefore, in neither case can one conclude that the string will initially start to break up.

Recently, in ref. [26] the dynamics of twisted vortex loops (vortons) has been studied in a model with two scalar fields. There, it has been found that such a loop is stable against axially symmetric perturbations (eg. expansion), and unstable against non-axially symmetric ones. Longer time evolution shows that the loop breaks up into smaller pieces. In our opinion however, this result in itself is not decisive to support a breakup scenario of twisted strings in the semilocal model. Besides the different initial dynamics, another reason why breakup of the string is unlikely is that, in order for the string to break up, string ends and droplet-like configurations must form. Clearly if a string breaks, its magnetic flux must end in something like a magnetic monopole resulting in a large energy contribution. We expect the situation to be rather different when non-Abelian gauge fields are present, since then a possible configuration for a string piece would be the famous dumbbell of Nambu [18].

Another interesting question examined in ref. [16] is the stability of solutions with periodic boundary conditions at $z = 0, L$ (this is an approximation of a vortex loop, if L is large). In the semilocal model, periodicity constrains the twist, so that ωL has to be an integer multiple of 2π ,

$$\omega L = 2\pi p, \quad p \in \mathbb{Z}. \tag{2.34}$$

In this case, for not very small values of ω (i.e. not very far from the bifurcation), the $k = \omega$ mode corresponds to a negative eigenvalue, therefore these short strings with periodic boundary conditions are unstable in the semilocal model.

An interesting observation of ref. [16] is that considering twisted strings as solutions of the $SU(2) \times U(1)$ (electroweak) gauge theory, the instability found in the semilocal model can be excluded by imposing periodic boundary conditions. In the electroweak model, with a suitable gauge transformation involving the electromagnetic and the third $SU(2)$ component vector fields, the z -dependence of the string can be gauged away and therefore periodic solutions of arbitrary length exist. However such periodic strings are *not gauge equivalent* anymore to a piece of a twisted string, the latter being a discontinuous configuration with infinite energy. Looking at the gauge transformation connecting the twisted and the untwisted string, the generator of this transformation is proportional to z . When z is made periodic, it ceases to be a univalued function, and thus, the vector field component proportional to dz ceases to be a pure gauge.

The gauge transformation which removes the twist of the vortex in the gauged $SU(2) \times U(1)$ model renders the homogeneously expanding mode z -dependent, and thus, this perturbed solution is excluded by the periodicity condition. This means that these small periodic strings are stable with respect to purely $U(1)$ perturbations, however, perturbations in the $SU(2)$ non-Abelian sector can still cause instabilities. Whether such instabilities exist, could only be decided considering all perturbations in the $SU(2) \times U(1)$ gauge theory, as in the case of electroweak strings [23]. Ref. [23] performs the linearized stability analysis of electroweak Z -strings. There, Z -strings are found to be unstable unless β is small or θ_W is large, and the instability mode is z -independent, occurs in the non-Abelian sector, and is decoupled from the other field components. However, these properties of the perturbation equations cannot be generalized to the case of the two-component strings of ref. [16].

3 Conclusion

We have found that twisted strings in the $SU(2)_{\text{global}} \times U(1)$ extended Abelian Higgs (semilocal) model possess a family of unstable modes, parametrized by their wave number in the z direction, $k \in [-k_m, k_m]$. This result is in agreement with that of ref. [16]. We have also obtained a semi-analytical description of twisted strings based on a suitable perturbative expansion near their bifurcation point with the embedded ANO solutions. The perturbative solution provides a surprisingly good quantitative approximation for the twisted strings themselves, as well as for the instability eigenvalues and eigenfunctions. This way the instability modes of twisted strings are obtained as deformations of those of the embedded ANO solutions.

Based on the similarity of the unstable modes of twisted and embedded ANO strings, we suggest that the initial time dynamics is also similar, i.e. at the outset of the instability twisted strings start to expand (nearly) homogeneously, just like embedded ANO ones. This idea is also supported by the fact, that the (homogeneous) lowest energy eigenmode of the embedded ANO string is also an energy lowering perturbation for twisted strings with ω not far from the bifurcation ω_b , although not an eigenmode. For values of the twist $\omega \ll \omega_b$, the formation of a growing lump leading to a local expansion of the string, is also an energy lowering perturbation. This suggests that for these strings (where the condensate in the core is already quite sizable), a local expansion is thus a plausible scenario for the initial dynamics of the instability. Twisted strings carry a global current. Although the initially homogeneous distribution of this current is changed into a z -dependent one, the current remains localized in leading order perturbation theory. Therefore the expansion of the string cannot proceed indefinitely, unless higher order radiation effects do indeed remove the current from the string.

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A Details of the perturbation equations

In this section we present some of the omitted details of the calculations. Although in other parts of the paper we have only studied the $n = 1$, $m = 0$ case, here we present the general, arbitrary n and m case.

A.1 The perturbation operator of the twisted vortices

Let us apply the Fourier transform (2.10) to the equations (2.7). A purely harmonic mode can be written in detail as

$$\begin{aligned}
 \delta\phi_1(z, t; x_i) &= e^{i(\Omega t - kz)} \delta\phi_1(k, \Omega; x_i) & \delta\phi_1^*(z, t; x_i) &= e^{i(\Omega t - kz)} \delta\phi_1^*(-k, -\Omega; x_i) \\
 \delta\phi_2(z, t; x_i) &= e^{i(\Omega t - (k - \omega)z)} \delta\phi_2(k, \Omega; x_i) & \delta\phi_2^*(z, t; x_i) &= e^{i(\Omega t - (k + \omega)z)} \delta\phi_2^*(-k, -\Omega; x_i) \\
 \delta A_\mu(z, t; x_i) &= e^{i(\Omega t - kz)} \delta A_\mu(k, \Omega; x_i)
 \end{aligned} \tag{A.1}$$

with the index i running over 1,2. The variables A_μ are real functions, therefore

$$A_\mu(k, \Omega, x_i) = A_\mu^*(-k, -\Omega, x_i). \tag{A.2}$$

Substituting these into equations (2.7) yields the perturbation operator \mathcal{M} of equation (2.11):

$$\mathcal{M} = \begin{pmatrix} \mathcal{D}_1 & \mathcal{U}_1 & \mathcal{V}_1 & \mathcal{V}'_1 & \mathcal{A}_1 & \mathcal{B}_1 & 0 \\ \mathcal{U}_1^* & \mathcal{D}_1^* & \mathcal{V}_1^* & \mathcal{V}'_1^* & \mathcal{A}_1^* & \mathcal{B}_1^* & 0 \\ \mathcal{V}_2 & \mathcal{V}'_2 & \mathcal{D}_2 & \mathcal{U}_2 & \mathcal{A}_2 & \mathcal{B}_2 & 0 \\ \mathcal{V}_2^* & \mathcal{V}'_2^* & \mathcal{U}_2^* & \mathcal{D}_2^* & \mathcal{A}_2^* & \mathcal{B}_2^* & 0 \\ \mathcal{A}_1^* & \mathcal{A}_1 & \mathcal{A}_2^* & \mathcal{A}_2 & \mathcal{D}_3 & 0 & 0 \\ \mathcal{B}_1^* & \mathcal{B}_1 & \mathcal{B}_2^* & \mathcal{B}_2 & 0 & \mathcal{D}_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{D}_3 \end{pmatrix} \tag{A.3}$$

with

$$\begin{aligned}
 \mathcal{D}_1 &= k^2 + \mathcal{D}_s + 2iA_i\partial_i + 2kA_3 + \mathcal{W}_1, \\
 \mathcal{D}_1^* &= k^2 + \mathcal{D}_s - 2iA_i\partial_i - 2kA_3 + \mathcal{W}_1, \\
 \mathcal{D}_2 &= (k - \omega)^2 + \mathcal{D}_s + 2iA_i\partial_i - 2(\omega - k)A_3 + \mathcal{W}_2, \\
 \mathcal{D}_2^* &= (k + \omega)^2 + \mathcal{D}_s - 2iA_i\partial_i - 2(\omega + k)A_3 + \mathcal{W}_2, \\
 \mathcal{D}_3 &= k^2 - \partial_i^2 + 2|\phi|^2,
 \end{aligned}$$

where

$$\mathcal{D}_s = -\partial_i^2 + A_i^2 + A_3^2 + \beta(|\phi|^2 - 1) \quad \mathcal{W}_i = (\beta + 1)\phi_i\phi_i^*$$

and

$$\begin{aligned}
 \mathcal{U}_1 &= (\beta - 1)\phi_1^2 & \mathcal{U}_2 &= (\beta - 1)\phi_2^2 \\
 \mathcal{V}_1 &= (\beta + 1)\phi_1\phi_2^* & \mathcal{V}_2 &= (\beta + 1)\phi_2\phi_1^* \\
 \mathcal{V}'_1 &= (\beta - 1)\phi_1\phi_2 & \mathcal{V}'_2 &= (\beta - 1)\phi_2\phi_1 \\
 \mathcal{A}_1 &= 2A_i\phi_1 + 2i\partial_i\phi_1 & \mathcal{A}_2 &= 2A_i\phi_2 + 2i\partial_i\phi_2 \\
 \mathcal{B}_1 &= 2A_3\phi_1 & \mathcal{B}_2 &= 2A_3\phi_2
 \end{aligned}$$

The Fourier-transform takes the equation (2.9) of the ghost modes into

$$\mathcal{D}_3\chi = \Omega^2\chi, \tag{A.4}$$

while a gauge transformation takes the form

$$\delta A_i \rightarrow \delta A_i + \partial_i\chi, \quad \delta A_3 \rightarrow \delta A_3 - ik\chi, \quad \delta A_0 \rightarrow \delta A_0 + i\Omega\chi. \tag{A.5}$$

It is useful to introduce complex coordinates

$$A_+ = \frac{e^{-i\vartheta}}{\sqrt{2}} \left(A_r - \frac{i}{r} A_\vartheta \right) \quad A_- = \frac{e^{i\vartheta}}{\sqrt{2}} \left(A_r + \frac{i}{r} A_\vartheta \right). \quad (\text{A.6})$$

Fourier expansion in the angle variable (in cylindrical coordinates $x^1 = r, x^2 = \vartheta$, see eq. (2.12)), omitting the sum over ℓ yields

$$\begin{aligned} \delta\phi_1(k, \Omega) &= s_{1,\ell} e^{i(n+\ell)\vartheta}, & \delta\phi_1^*(-k, -\Omega) &= s_{1,-\ell}^* e^{-i(n-\ell)\vartheta}, \\ \delta\phi_2(k, \Omega) &= s_{2,\ell} e^{i(m+\ell)\vartheta}, & \delta\phi_2^*(-k, -\Omega) &= s_{2,-\ell}^* e^{-i(m-\ell)\vartheta}, \\ \delta A_+(k, \Omega) &= i a_\ell e^{i(\ell-1)\vartheta}, & \delta A_-(-k, -\Omega) &= -i a_{-\ell}^* e^{i(\ell+1)\vartheta}, \\ \delta A_0(k, \Omega) &= a_{0,\ell} e^{i\ell\vartheta}, & a_{0,-\ell}^*(-k, -\Omega) &= a_{0,\ell}(k, \Omega), \\ \delta A_3(k, \Omega) &= a_{3,\ell} e^{i\ell\vartheta}, & a_{3,-\ell}^*(-k, -\Omega) &= a_{3,\ell}(k, \Omega). \end{aligned} \quad (\text{A.7})$$

Substituting this into the equations of motion, (2.7) assumes the form of the eigenvalue problem (2.13) with the operator

$$M_\ell = \begin{pmatrix} D_1 & U_1 & V & V' & A_1 & A'_1 & B_1 & 0 \\ U_1 & D_1^* & V' & V & A'_1 & A_1 & B_1 & 0 \\ V & V' & D_2 & U_2 & A_2 & A'_2 & B_2 & 0 \\ V' & V & U_2 & D_2^* & A'_2 & A_2 & B_2 & 0 \\ A_1 & A'_1 & A_2 & A'_2 & D_3 & 0 & 0 & 0 \\ A'_1 & A_1 & A'_2 & A_2 & 0 & D_3^* & 0 & 0 \\ B_1 & B_1 & B_2 & B_2 & 0 & 0 & D_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_4 \end{pmatrix} \quad (\text{A.8})$$

with

$$\begin{aligned} D_1 &= D_s + \frac{(n(1-a) + \ell)^2}{r^2} + k^2 + 2k\omega a_3 + W_1 \\ D_1^* &= D_s + \frac{(n(1-a) - \ell)^2}{r^2} + k^2 - 2k\omega a_3 + W_1 \\ D_2 &= D_s + \frac{(m - na + \ell)^2}{r^2} + (k - \omega)^2 + 2(k - \omega)\omega a_3 + W_2 \\ D_2^* &= D_s + \frac{(m - na - \ell)^2}{r^2} + (k + \omega)^2 - 2(k + \omega)\omega a_3 + W_2 \\ D_3 &= D_a + \frac{(\ell - 1)^2}{r^2} \\ D_3^* &= D_a + \frac{(\ell + 1)^2}{r^2} \\ D_4 &= D_a + \frac{\ell^2}{r^2} \end{aligned} \quad (\text{A.9})$$

with

$$\begin{aligned} D_s &= -\nabla_r^2 + \omega^2 a_3^2 + \beta(|f|^2 - 1), \\ W_i &= (\beta + 1)f_i^2, \\ D_a &= -\nabla_r^2 + k^2 + 2|f|^2 \end{aligned}$$

and

$$\begin{aligned}
 U_1 &= (\beta - 1)f_1^2 & U_2 &= (\beta - 1)f_2^2 \\
 V &= (\beta + 1)f_1f_2 & V' &= (\beta - 1)f_1f_2 \\
 A_1 &= -\sqrt{2}\left(f_1' - \frac{nf_1}{r}(1-a)\right) & A_2 &= -\sqrt{2}\left(f_2' - \frac{m-na}{r}f_2\right) \\
 A_1' &= \sqrt{2}\left(f_1' + \frac{nf_1}{r}(1-a)\right) & A_2' &= \sqrt{2}\left(f_2' + \frac{m-na}{r}f_2\right) \\
 B_1 &= 2\omega a_3 f_1 & B_2 &= 2\omega(a_3 - 1)f_2.
 \end{aligned}$$

The expansion of the gauge transformation generator function can be chosen as

$$\chi = \chi_\ell e^{i\ell\theta}. \quad (\text{A.10})$$

Using this expansion, the ghost mode equation (A.4) assumes the form

$$D_4\chi_\ell = \Omega^2\chi_\ell. \quad (\text{A.11})$$

Gauge transformations satisfying the above equation act on the fields as

$$\begin{aligned}
 s_{a,\ell} &\rightarrow s_{a,\ell} + i\chi_\ell f_a, & s_{a,-\ell}^* &\rightarrow s_{a,-\ell}^* - i\chi_\ell f_a, \\
 a_\ell &\rightarrow a_\ell - \frac{i}{\sqrt{2}}\left(\chi_\ell' + \frac{\ell\chi_\ell}{r}\right), & a_{-\ell}^* &\rightarrow a_{-\ell}^* + \frac{i}{\sqrt{2}}\left(\chi_\ell' - \frac{\ell\chi_\ell}{r}\right), \\
 a_{3,\ell} &\rightarrow a_{3,\ell} - ik\chi_\ell, & a_{0,\ell} &\rightarrow a_{3,\ell} + i\Omega\chi_\ell.
 \end{aligned} \quad (\text{A.12})$$

A.2 The $\beta \rightarrow \infty$ limiting case

In the $\beta \rightarrow \infty$ case the linearized form of the constraint $\phi_a^*\phi_a = 1$ is

$$\phi_a^*\delta\phi_a + \phi_a\delta\phi_a^* = 0. \quad (\text{A.13})$$

This can be taken into account by the substitution

$$\begin{aligned}
 s_{1,\ell} &= t_\ell \cos\theta + t_{1,\ell}, & s_{1,-\ell}^* &= t_\ell \cos\theta - t_{1,\ell}, \\
 s_{2,\ell} &= -t_\ell \sin\theta + t_{2,\ell}, & s_{2,-\ell}^* &= -t_\ell \sin\theta - t_{2,\ell},
 \end{aligned} \quad (\text{A.14})$$

and the equations of t_ℓ , $t_{1,\ell}$ and $t_{2,\ell}$ can be obtained with a somewhat lengthy but straightforward calculation. Here we present only the $\ell = 0$ case:

$$\frac{1}{r}(rt_0')' = \frac{A_{t0}}{r^2}t_0 + D_{t0}t_0 + E_{t0}t_1 + F_{t0}t_2 - \frac{m-n}{\sqrt{2}r}\sin(2\theta)(a_0 + a_0^*) + \omega\sin(2\theta)a_{3,0} \quad (\text{A.15})$$

with $A_{t0} = -(m-n)(m+n-2na)\cos(2\theta)$, $D_{t0} = k^2 - \Omega^2 - \omega^2(1-2a_3)\cos(2\theta)$, $E_{t0} = 2k\omega a_3 \cos\theta$ and $F_{t0} = 2k\omega(1-a_3)\sin\theta$. Similarly

$$\frac{1}{r}(rt_1')' = \frac{A_{t1}}{r^2}t_1 + C_{t1}t_1 + D_{t1}t_0 + E_{t1}t_2 - \sqrt{2}\cos\theta\theta'(a_0 - a_0^*) \quad (\text{A.16})$$

with $A_{t1} = -\frac{1}{2} [m^2 - n^2 - 4(m-n)na \cos^2 \theta + (m^2 - n^2) \cos(2\theta)]$, $C_{t1} = 1 + k^2 - \Omega^2 - \frac{\omega^2}{2} - \cos(2\theta) + 2\omega^2 a_3 \cos^2 \theta - \frac{\omega^2}{2} \cos(2\theta) - (\theta')^2$, $D_{t1} = 2k\omega a_3 \cos \theta$ and $E_{t1} = \sin(2\theta)$, and

$$\frac{1}{r}(rt'_2)' = \frac{A_{t2}}{r^2}t_2 + C_{t2}t_2 + D_{t2}t_0 + E_{t2}t_1 + \sqrt{2} \sin \theta \theta' (a_0 - a_0^*) \quad (\text{A.17})$$

with $A_{t2} = -\frac{1}{2} [n^2 - m^2 + 4(m-n)na \cos^2 \theta + (m^2 - n^2) \cos(2\theta)]$, $C_{t2} = 1 + k^2 - \Omega^2 + \frac{\omega^2}{2} + \cos(2\theta) - 2\omega^2 a_3 \sin^2 \theta - \frac{\omega^2}{2} \cos(2\theta) - (\theta')^2$, $D_{t2} = -2k\omega(a_3 - 1) \sin \theta$ and $E_{t2} = \sin(2\theta)$.

A.3 The expansion of the perturbation operator in the bifurcation parameter

Let us first expand the components of the perturbation matrix in eq. (A.8) to second order in ϵ . Here we present only the terms used later on in this paper. The differential operators expand as

$$\begin{aligned} D_2^{(0)} &= -\nabla_r^2 + \frac{(m - na^{(0)} + \ell)^2}{r^2} + (k - \omega_b)^2 + \beta((f_1^{(0)})^2 - 1), \\ D_2^{(1)} &= 0, \\ D_2^{(2)} &= 2\omega_b\omega_2 - 2k\omega_2 - \frac{2(m - na^{(0)} + \ell)na^{(2)}}{r^2} + 2\omega_b(k - \omega_b)a_3^{(2)} \\ &\quad + 2\beta f_1^{(0)} f_1^{(2)} + (2\beta + 1)(f_2^{(1)})^2, \end{aligned} \quad (\text{A.18})$$

$$\begin{aligned} D_2^{*(0)} &= -\nabla_r^2 + \frac{(m - na^{(0)} - \ell)^2}{r^2} + (k + \omega_b)^2 + \beta((f_1^{(0)})^2 - 1), \\ D_2^{*(1)} &= 0, \\ D_2^{*(2)} &= 2\omega_b\omega_2 + 2k\omega_2 - \frac{2(m - na^{(0)} - \ell)na^{(2)}}{r^2} - 2\omega_b(k + \omega_b)a_3^{(2)} \\ &\quad + 2\beta f_1^{(0)} f_1^{(2)} + (2\beta + 1)(f_2^{(1)})^2, \end{aligned} \quad (\text{A.19})$$

Scalar-scalar interaction terms:

$$\begin{aligned} V^{(0)} &= 0, & V^{(1)} &= (\beta + 1)f_1^{(0)} f_2^{(1)}, & V^{(2)} &= 0, \\ V'^{(0)} &= 0, & V'^{(1)} &= (\beta - 1)f_1^{(0)} f_2^{(1)}, & V'^{(2)} &= 0. \end{aligned} \quad (\text{A.20})$$

Scalar-vector interaction terms:

$$\begin{aligned} A_2^{(1)} &= -\sqrt{2} \left(f_2^{(1)'} - \frac{m - na^{(0)}}{r} f_2^{(1)} \right), \\ A_2'^{(1)} &= \sqrt{2} \left(f_2^{(1)'} + \frac{m - na^{(0)}}{r} f_2^{(1)} \right), \end{aligned} \quad (\text{A.21})$$

while $A_2^{(0)} = A_2'^{(0)} = A_2^{(2)} = A_2'^{(2)} = B_2^{(0)} = B_2^{(2)} = 0$, and

$$B_2^{(1)} = -2\omega_b f_2^{(1)}. \quad (\text{A.22})$$

This ϵ -expansion of the perturbation matrix elements can be used to obtain the unstable eigenvalue of twisted vortices close to the bifurcation point via perturbation series.

β	$f^{(1)}$	$a^{(2)}$
1.25	0.92418	0.53485
2	1.09935	0.61657
2.5	1.19677	0.65959

Table 7. Local vortex background data

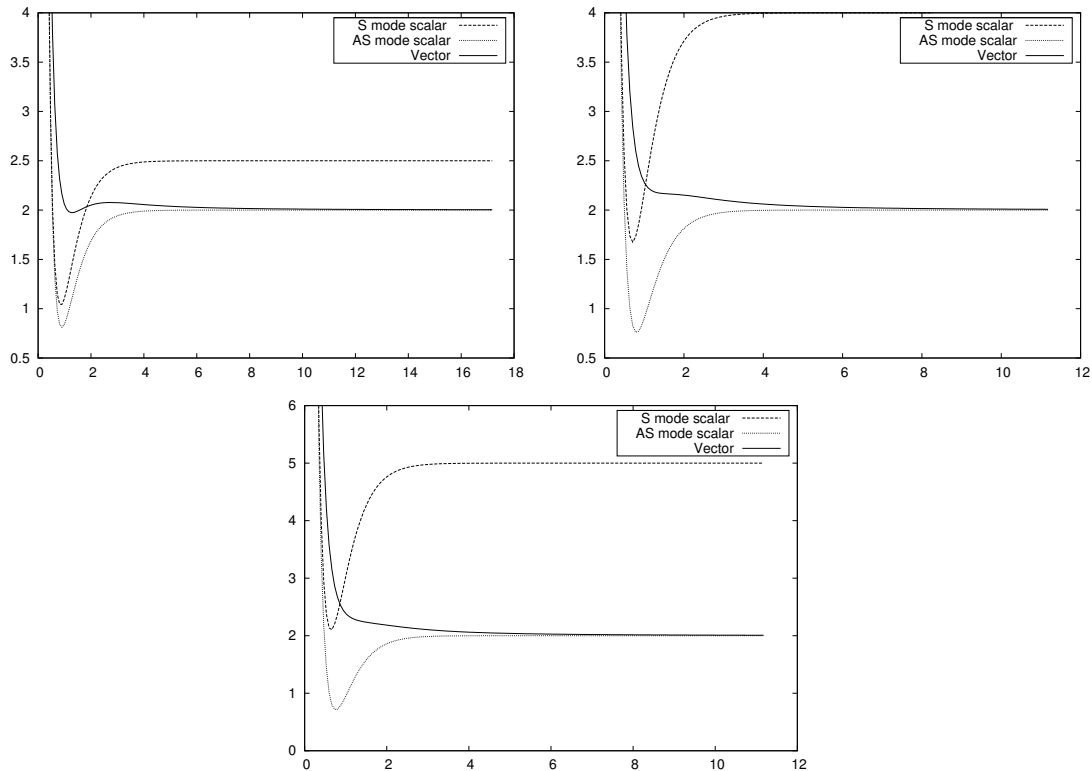


Figure 10. Potentials in the ANO vortex perturbation equations for $\beta = 1.25$ (left), $\beta = 2$ (right) and $\beta = 2.5$ (below)

B Perturbations of the ANO vortices

The equations describing local vortices can be obtained from the twisted vortex equations (1.9) by taking $f_2 = a_3 = 0$. With the same numerical methods as used in the case of the twisted vortices, local vortex background data can be obtained, see table 7. Here, the $f^{(1)}$ and $a^{(2)}$ is defined by the behavior of the vortex profile functions at the origin, $f \sim f^{(1)}r + O(r^2)$ and $a \sim a^{(2)}r^2 + O(r^4)$ (shooting parameters).

The perturbations of local vortices can be described using equation (A.8) with $\omega = \omega_b$ and $f_2 = a_3 = 0$. The $a_{3\ell}$ component decouples, and it's equation becomes that of the ghost modes, eq. (A.11). The $s_{2,\ell}$ and $s_{2,-\ell}^*$ components also decouple; their equation (1.14) has already been given in section 1. The instabilities arising from the extension of the model with a second Higgs field are described by these equations. For further details see [3, 9]. In the $\ell = 0$ case, there is a further decoupling in the $(s_{1,0}, s_{1,0}^*, a_0, a_0^*)$ sector: symmetric: $\psi_S =$

	Symm. mode	Anti. mode	a_3 mode
β	λ_S^2	λ_A^2	$\lambda_{a_3}^2$
1.25	1.82404	1.62442	1.62442
2		1.76100	1.76100
2.5		1.81813	1.81813

Table 8. Parameters for the $\ell = 0$ ANO vortex perturbation modes for $\beta = 1.25, 2$ and 2.5

(s_1, s_1, a_0, a_0) and antisymmetric $\psi_A = (s_1, -s_1, a_0, -a_0)$ modes are decoupled. For values of $\beta < 1.5$ there is a bound mode in each of these two sectors. For $\beta > 1.5$ the symmetric mode becomes quasi-bound. The parameters for these modes (normed in a way that $s'_1(0) = 1$) are summarized in table 8. There, λ_{A,S,a_3}^2 denotes the lowest eigenvalue (corresponding to $k = 0$). Figure 10 shows the potential terms for s_0 and a_0 in the symmetric and antisymmetric sectors, respectively, for $\beta = 1.25, 2$ and 2.5 . These explain it clearly why the mode ψ_S becomes quasi-bound for higher β (i.e. for $\beta > 1.5$, there is no bound mode in the symmetric sector, but for $2 < \Omega^2 < 2\beta$, the scalar channel of the scattering problem is closed).

The gauge dependence of the eigenmodes and the relation of ghost modes to these is rather straightforward in the case of the ANO vortices. In the analysis of the spectrum of these vortices, in ref. [11] it has been noted that if one considers the perturbation problem of the vortices in 2+1D (i.e. z -independent perturbations), then the ghosts cancel the perturbations of A_0 , and some modes of the gauge fixed perturbation operators. If we allow z -dependent perturbations, instead of δA_0 a combination of δA_0 and δA_3 is canceled. Ref. [11] also identifies the non-physical mode with the antisymmetric mode in the $\ell = 0$ case, based on the small difference in their numerical eigenvalues. Here, we would like to supplement this identification of the canceled mode with some analytical calculation. Examining eqs. (A.12), one can easily conclude that in the symmetric case, the gauge generator drops out of $s_1 = (s_{1,0} + s_{1,0}^*)/2$ and $a_0 = (a_0 + a_0^*)/2$. On the contrary, such a cancellation does not happen in the antisymmetric case, thus the ghost eigenvalues shall appear in the spectrum of the antisymmetric mode. On a twisted vortex background, the couplings are more complicated, and therefore such a straightforward analysis is not possible.

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